Prop: $K \leq H \leq G$.

Let $\{x_i\}$ be a set of (left) coset representatives of K in H, $\{y_j\}$ a set of (left) coset representatives of H in G. Then $\{y_j x_i\}$ is a set of (left) coset rep. of K in G. Consequently, we have

$$\underline{(G:K) = (G:H)\cdot(H:K)}\text{。}$$

pf:

$$\left.\begin{aligned} H &= \coprod_{i\in I} x_i \cdot K \\ G &= \coprod_{j\in J} y_j \cdot H \end{aligned}\right\} \Rightarrow$$

$$G = \bigcup_{\substack{i\in I\\ j\in J}} y_j x_i K$$

It remains to show

$$y_{j'} x_{i'} K = y_j x_i K \Leftrightarrow y_j = y_{j'},\ x_i = x_{i'}.$$

Since $x_{i'}, x_i, K \subseteq H$,

$$y_{j'} x_{i'} K = y_j x_i K \Rightarrow \underset{\substack{\| \\ y_{j'}\cdot H}}{y_{j'} x_{i'} K\cdot H} = \underset{\substack{\| \\ y_j H}}{y_j x_i K\cdot H}$$

$$\Rightarrow \quad y_{j'} = y_j$$

Thus, we have

$$y_j x_{i'} K = y_j x_i K$$

$$\Rightarrow \quad y_j^{-1} \cdot y_j \, x_i' K = y_j^{-1} y_j x_i K$$

$$\Rightarrow \quad x_i' \cdot K = x_i K \quad \Rightarrow x_i' = x_i.$$

#

Lecture 3. Normal Subgroup.

Observation: $f: G \to G'$ homo.

The subgp $\mathrm{Ker}(f) \le G$ has the following property:

$\forall x \in G$

$\underline{x \cdot \mathrm{Ker}(f) \cdot x^{-1} = \mathrm{Ker}(f)}$

or equivalently

$x \cdot \mathrm{Ker}(f) = \mathrm{Ker}(f) \cdot x$

Pf: $f(x \,\mathrm{Ker}(f)\, x^{-1}) = f(x) \cdot f(\mathrm{Ker}(f)) f(x)^{-1}$

$= f(x)\, e\, f(x)^{-1} = e$

$\Rightarrow x \,\mathrm{Ker}(f)\, x^{-1} \subseteq \mathrm{Ker}(f)$.

Thus $x^{-1} \mathrm{Ker}(f)\, x \subseteq \mathrm{Ker}(f)$

$\Rightarrow \mathrm{Ker}(f) \subseteq x \,\mathrm{Ker}(f)\, x^{-1}$

$\Rightarrow$ $\mathrm{Ker}(f) = x \,\mathrm{Ker}(f)\, x^{-1}$

Def (normal subgp)

A subgp $H \le G$ is called normal if, $\underline{\forall x \in G}$

$\underline{xHx^{-1} = H}$ ($\Leftrightarrow$ $\underline{xH = Hx}$) is satisfied.

Notation: $H \triangleleft G$ (normal subgp)

CLAIM: Show each normal subgp is the kernel of some homo!

Step 1: Group structure on the set of cosets.

Let G' be the set of (left = right) cosets. i.e.

$$G' = \{xH \mid x \in G\} = \{Hx \mid x \in G\}$$

Notice:

$$(xH)(yH) = x(Hy)H$$

$$= x(yH)H$$

use the normal cond! $= (xy)(H \cdot H)$

$$= xy \cdot H$$

Define

$$\begin{array}{ccc} G' \times G' & \longrightarrow & G' \\ \in & & \in \\ (xH, yH) & \longmapsto & xyH \end{array}$$

check: (i) associative

(ii) $e_{G'} = eH = H$

(iii) $(xH)^{-1} = x^{-1}H$

Def (factor group)

For each $H \triangleleft G$, we define the factor group of G by H (denoted by G/H) to be the above G', together with the above group structure.

Step 2 Canonical homomorphism

$$\begin{array}{ccc} G & \xrightarrow{f} & G/H \\ \in & & \in \\ x & \longmapsto & xH \doteq \bar{x} \end{array}$$

Check: (i) f is surjective

(ii) f is group homo

(iii) $\ker(f) = f^{-1}(\{H\}) = H$

In conclusion: $\forall\, H \triangleleft G$,

$$H = \ker(f : G \to G/H)$$

Remarks: (1) $H_i \triangleleft G$, $i \in I$

$$\Rightarrow \bigcap_{i \in I} H_i \triangleleft G \qquad \text{(Exercise)}$$

(2) (normalizer and centralizer)

For $S \subseteq G$, define

$$N_S \triangleq \{x \in G \mid xSx^{-1} = S\}$$

check: $N_S \leq G$. N_S is called the normalizer of S in G.

~~For~~ $Z_S \triangleq \{x \in G \mid xyx^{-1} = y, \forall y \in S\}$

$$\Updownarrow$$

$$xy = yx$$

check: $Z_S \leq G$. Z_S is called the centralizer of S in G.

Obviously: For $S = \{x\}$., $N_S = Z_S$.

Def (Center)

The center of G is defided to Z_G, denoted sometimes by $Z(G)$.

(Exercise):

Clearly, $G = Z(G) \iff G$ is abelian.

In general $Z(G) \lhd G$.

Examples: (1) $\det: GL_n(k) \longrightarrow GL_1(k) = k^\times$

$$A \longmapsto \det(A)$$

Recall:

For $A = (a_{ij})_{n\times n}$ $\det(A) \triangleq \frac{1}{n!}\sum_{\sigma\in S_n} \mathrm{Sgn}(\sigma)\, a_{1\sigma(1)} \cdots a_{n\sigma(n)}$

Then det is a homo.

$$SL_n(K) \triangleq \ker(\det)$$
$$= \{A \in GL_n(K) \mid \det(A) = 1\}$$

<u>Special linear group</u>

(2) (The group of affine $\underset{\text{linear}}{\ }$ transformations)

$\forall A \in GL_n(K)$, $b \in K^n$, define

$$\begin{aligned} T_{A,b} : K^n &\longrightarrow K^n \\ v &\longmapsto A \cdot v + b \end{aligned}$$

$$G = \{T_{A,b} \mid A \in GL_n(K),\ b \in K^n\}$$

$(G, \circ)$ = group of <u>affine linear transformations</u> of K^n.

Def:
$$\begin{aligned} G &\xrightarrow{f} GL_n(K) \\ T_{A,b} &\longmapsto A \end{aligned}$$

Check: $f(T_{A',b'} \circ T_{A,b}) = f(T_{A',b'}) \cdot f(T_{A,b})$

$$\parallel$$

$$f(T_{A'A,\ A'b+b'})$$

$$\begin{array}{ccc} \ker(f) = \{ T_{Id,b} \mid b \in K^n \} & \simeq & K^n \quad \text{(check the isomorphism)} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ T_{Id,b} & \longmapsto & b \end{array}$$

Note: Set $N = \ker(f) \trianglelefteq G$

$H = \{ T_{A,0} \mid A \in GL_n(K) \} \leq G$.

Then (i) $G = N \cdot H = H \cdot N$

(ii) $N \cap H = \{ \underset{T_{Id,0}}{\underset{\shortparallel}{Id}} \}$

But $G \not\simeq N \times H$..

G is a semi-product of H and N.

Semi-product is a very important example of group!

Exercise: (1) $H \leq G$. Then N_H is the largest subgp of G such that H is normal in this subgroup. That is,

If $K \leq G$, $H \trianglelefteq K$, then $K \leq N_H$.

(2) $\forall K \leq N_H$, $K \cdot H = H \cdot K$ is a subgroup (of N_H)

Moreover, $H \trianglelefteq K \cdot H$

For $H \triangleleft G$, and $x, y \in G$. If $xH = yH$ (or $\bar{x} = \bar{y}$),

then we write

$$x \equiv y \pmod{H}$$

(say x and y are congruent modulo H).

If G is abelian, then

$$x \equiv y \pmod{H} \iff x - y \equiv 0 \pmod{H}$$

Example: $\left(\mathbb{Z}/n\mathbb{Z} = \{[0], \dots, [n-1]\}, +\right)$

where $[i] = \{x \in \mathbb{Z} \mid x \equiv i \pmod{n}\}$

$$x \equiv i \pmod{n} \iff n \mid x - i \iff x - i \in n\mathbb{Z}$$

Can. homo: $(\mathbb{Z}, +) \longrightarrow (\mathbb{Z}/n\mathbb{Z}, +)$

$$x \longmapsto \bar{x}$$

Note $\{0, 1, \dots, n-1\}$ is just a set of coset rep for $H = n\mathbb{Z}$ in $\mathbb{Z}$.

Exact sequence:

Let $G' \xrightarrow{f} G \xrightarrow{g} G''$

be a sequence of homo $\Leftrightarrow$ $f: G' \to G'$ both are homo.
$g: G \to G''$

Def: (exact sequence)

The above sequence is exact if

$\mathrm{Im}(f) = \mathrm{Ker}(g)$.

The equality is an equality of two subgroups in G.

Example: For $H \lhd G$, we have a canonical exact sequence

$$H \xhookrightarrow{j} G \xrightarrow{\pi} G/H,$$

where $j: H \hookrightarrow G$ the ~~natu~~ inclusion.

In general, for a sequence of form

$$G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \cdots \xrightarrow{f_{n-1}} G_n,$$

it is exact, if it is exact at each joint, that is

$$\mathrm{Im}(f_i) = \mathrm{Ker}(f_{i+1}), \quad i = 1, \cdots, n-2$$

Thus, the above can. exact sequence extends to the following exact sequence:

$$0 \longrightarrow H \xrightarrow{i} G \xrightarrow{\pi} G/H \longrightarrow 0 .$$

where 0 = the trivial group, $0 \to H$, $G/H \to 0$ are the canonical homomorphisms.

Prop ("fundamental thm of group theory")

Any group homo $f: G \longrightarrow G'$ induces a canonical isomorphism

$$G/\ker(f) \overset{f_*}{\simeq} \mathrm{Im}(f).$$

pf: Step 1 (Construction of f_*).

We're trying to define

$$f_* : G/\ker(f) \longrightarrow \mathrm{Im}(f)$$
$$\bar{x} \longmapsto f(x)$$

But, we need to verify the well-definedness of f_*! Namely,

$\bar{x} = \bar{x}' \implies f_*(\bar{x}) = f_*(\bar{x}')$ (Exercise).

Step 2 (f_* is an isomorphism)

(i) f_* is gp homo.

$$\begin{array}{ccc} f_*(\bar{x}\cdot\bar{y}) & \overset{?}{=} & f_*(\bar{x})\, f_*(\bar{y}) \\ \| & & \| \\ f_*(\overline{xy}) & & f(x)\, f(y) \\ \| & & \\ f(xy) & \checkmark & \end{array}$$

(ii) f_* inj.

$$f_*(\bar{x}) = f(x) = e$$

$$\Rightarrow x \in \mathrm{Ker}(f) \Rightarrow \bar{x} = \mathrm{Ker} f = \bar{e} \in G/\mathrm{Ker}(f)$$

(iii) f_* surj

Obvious.

Thus f_* is an isomorphism. $\cancel{\Leftrightarrow \mathrm{Ker}(f_*) = \{}$

Variants: (1) $H \leqslant G$. Set $\widetilde{H} = \bigcap_{\substack{N \lhd G \\ H \leqslant N}} N$ (the smallest normal subgp of G containing H)

Sometimes, we call $\widetilde{H}$ the normal closure of H in G.

In particular if $H \triangleleft G$, then $\tilde{H} = H$.

Prop: $f: G \to G'$ homo with $H \leq \ker(f)$.

Then $\tilde{H} \leq \ker(f)$, and f induces canonically a Comm. diagram of homo.

$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr, "\pi"'] & \circlearrowright & G' \\ & G/\tilde{H} \arrow[ur, "f_*"'] & \end{tikzcd}$$

pf: Exercise. #

If $H = \ker(f)$, then $f_*: G/\ker(f) \to G'$ is the can. homo as above.

(2) Suppose $H \triangleleft G$, $K \triangleleft G$, and $K \leq H$, then we have a <u>can. isomorphism</u>

$$(G/K)\big/(H/K) \xrightarrow{\sim} G/H$$

pf: Since $K \leq H$, we have canonically

$$\begin{array}{ccc} G/K & \xrightarrow{\varphi} & G/H \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ xK & \longmapsto & xH \end{array}$$

Certainly φ is a surjective homomorphism.

$\ker(\varphi) = ?$

$$\varphi(xK) = e = H.$$

$$\Leftrightarrow \quad xH = H$$

$$\Leftrightarrow \quad x \in H$$

Thus, $\ker(\varphi) = \{xK \mid x \in H\}$

$$= H/K.$$

Thus, $$(G/K)\big/(H/K) \xrightarrow{\varphi_*} G/H.$$

#

We have a beautiful comm. diagram of exact sequences, to summarize the above proof:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K \arrow[r, "id"] \arrow[d] & K \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H \arrow[r] \arrow[d, "can"] & G \arrow[r] \arrow[d, "can"] & G/H \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & H/K \arrow[r] \arrow[d] & G/K \arrow[r] \arrow[d] & G/H \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

(3). $H, K \leq G$, Assume $K \leq N_H$, Then

$$\begin{cases} HK = KH \leq G \\ H \cap K \lhd K. \end{cases}$$

and a can. isomorphism

$$HK/H \xrightarrow{\sim} K/H\cap K$$

Pf: It is obvious to have

$$HK \leq G, \quad H \cap K \lhd K.$$

Consider the composite of homo

$$\begin{array}{ccccc} \varphi: K & \hookrightarrow & HK & \twoheadrightarrow & HK/H \\ \in & & \in & & \in \\ x & \longmapsto & Hx & \longmapsto & \overline{Hx} \end{array}$$

Clearly: φ is surjective

$\mathrm{Ker}(\varphi) = ?$

$$\left.\begin{array}{r} \varphi(x) = e = H \Leftrightarrow Hx = H \Leftrightarrow x \in H \\ x \in K \end{array}\right\} \Leftrightarrow x \in H \cap K.$$

Thus $\mathrm{Ker}(\varphi) = H \cap K$, and

$$K/H\cap K \xrightarrow[\sim]{\varphi_*} HK/H.$$

#

(4) $f: G \longrightarrow G'$ homo

$H' \lhd G'$. Set $H = f^{-1}(H')$. Then

$H \lhd G$, and we have a can. inj homo:

$$G/H \xhookrightarrow{f_*} G'/H'$$

Pf: Direct check $H \lhd G$ (Rmk: if $H' \leq G'$, then $f^{-1}(H') \leq G$)

Consider the composite

$$\varphi: G \xrightarrow{f} G' \xrightarrow{\pi} G'/H'$$

$\ker(\varphi) = ?$

$$\varphi(x) = e = H' \iff f(x) \in H'$$
$$\iff x \in f^{-1}(H') = H$$

Thus $\ker(\varphi) = H$

$$\Rightarrow \quad \begin{tikzcd} G/H \arrow[r, "\sim"] \arrow[rr, bend right, "f_*"'] & \operatorname{Im}(\varphi) \arrow[r, hook] & G'/H' \end{tikzcd}.$$

#

Example: $H_1, H_2 \leq G$. Assume $(G : H_i) < +\infty$, $i = 1, 2$

Show $(G : H_1 \cap H_2) < +\infty$.